MyJ

Chapter 3

1. $A \subset \mathbb{R}^n$, $m^*(A)=0$. WTS $\forall B \subset \mathbb{R}^n$, $m^*(A\cup B) = m^*(B)$

Pf: $m^*(A\cup B) \le m^*(A) + m^*(B) = m^*(B) \le m^*(A\cup B)$ $\square$

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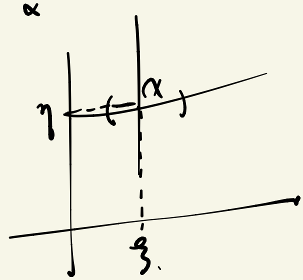
2. $\overset{\subseteq \mathbb{R}^2}{E} = \{\alpha = (\xi,\eta) : \xi,\eta$ 至少有一个是有理数$\}$. $m^*(E) = ?$

Pf: $E \Leftrightarrow \mathbb{R}^2$ 中可列条直线. $m^*(l_\alpha) = 0$

$m^*(E) = m^*(\bigcup_{\alpha\in I} l_\alpha) \le \sum_\alpha m(l_\alpha) = 0$

也可不妨设 $\xi \in \mathbb{Q}$. $\eta \notin \mathbb{Q}$.



$\exists$ 任意小 $\varepsilon > 0$

$\left(\xi_i - \frac{\varepsilon}{2^i}, \xi_i + \frac{\varepsilon}{2^i}\right) = U_i$

cover l_α ? $\square$

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5. 设 $A_1, A_2 \subset \mathbb{R}^n$, $A_1 \subset A_2$. A_1 is a measurable set and $m(A_1) = m^*(A_2) < +\infty$. WTS A_2 is measurable

Pf: A_i is measurable iff $\forall E \subset \mathbb{R}^n$, $m^*(E) = m^*(E\cap A_i) + m^*(E\cap A_i^c)$

Take $E = A_2$, $i=1$ (v).

$+\infty > \underset{\sim\sim}{m^*(A_2)} = m^*(\underbrace{A_2 \cap A_1}_{A_1}) + m^*(\underline{A_2\cap A_1^c})$

$= \underset{\sim\sim}{m(A_1)} + m^*(A_2 - A_1)$

$\Rightarrow m^*(A_2 - A_1) = 0$

$A_2 = A_1 \cup (A_2 - A_1) \in \mathcal{L}$ $\square$

6. $\alpha \in \mathbb{R}^n$, $\tau_\alpha x = x + \alpha$. WTS: 在平移变换下. $\forall E \subset \mathbb{R}^n$

$$m^*(E) = m^*(\tau_\alpha E).$$

Pf: Take an open cover $I_k = (a_k, b_k)$

$$\tau_\alpha I_k = (a_k + \alpha, b_k + \alpha)$$

By definition $\sum |I_k| = \sum |\tau_\alpha I_k| \geqslant m^*(\tau_\alpha E)$

$$\therefore m^*(E) = \inf \sum |I_k| \geqslant m^*(\tau_\alpha E).$$

On the other hand

$$m^*(\underbrace{\tau_\alpha E}_{E'}) \geqslant m^*(\tau_{-\alpha} \underbrace{\tau_\alpha E}_{E'}) = m^*(E). \qquad \square$$

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9. WTS. $\forall A \subset \mathbb{R}^p$, $B \subset \mathbb{R}^q$. we have $m^*(A \times B) \leq m^*(A) \cdot m^*(B)$.

Pf. By definition, $\exists (Q_i)_{i \geq 1}$ $(I_j)_{j \geq 1}$. s.t $A \subset \bigcup_{i \geq 1} Q_i$

$B \subset \bigcup_{j \geq 1} I_j$

$$m^*(A \times B) \leq \sum_{i,j} |Q_i \times I_j|$$

$$\leq \left(\sum_i |Q_i|\right) \cdot \left(\sum_j |I_j|\right)$$

$$\inf \longrightarrow \leq m^*(A) \cdot m^*(B). \qquad \square$$

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10. If $E \in \mathcal{L}$, $\forall \varepsilon > 0$, $\exists G$ open, F closed. s.t. $F \subset E \subset G$.

$$m(G - E) < \varepsilon. \quad m(E - F) < \varepsilon.$$

Pf: $\forall \varepsilon > 0$. $\exists$ open set $G \supset E$. s.t $m(G \setminus E) < \varepsilon$.

F closed. $\parallel$

$G \supset E^c$ s.t $m(G \setminus E^c) = m(G \cap E^c) = m(E \setminus G^c) < \varepsilon$

Further $\square$

11. $E \subset \mathbb{R}^q$, $\exists$ two serise measurable sets $\{A_n\}$. $\{B_n\}$

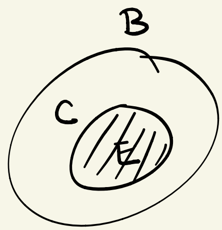
s.t $A_n \subset E \subset B_n$. & $m(B_n \setminus A_n) \to 0$ $(n \to +\infty)$ WTS E可测

Pf: Take $A = \bigcup_{n \geq 1} A_n$. $B = \bigcap_{n \geq 1} B_n \Rightarrow A . B$ are measurable.

and $A \subset E \subset B$.

$$B \setminus A = \bigcap_n B_n \cap \Big(\bigcup_{n \geq 1} A_n\Big)^c = \bigcap_{n \geq 1} (B_n \setminus A_n) \subset B_n \setminus A_n$$

$$\therefore m(\underbrace{B \setminus A}_{\text{null set}}) \leq m(B_n \setminus A_n) \longrightarrow 0 \ (n \to +\infty)$$



$\because B \setminus E \subset B \setminus A \quad \therefore C = B \setminus E$ is null set

$\Rightarrow E = B \setminus C$ is measurable. $\square$

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12. $E \subseteq \mathbb{R}^n$. If $\forall \varepsilon > 0$. $\exists$ closed set $F \subset E$ s.t $m^*(E - F) < \varepsilon$.

WTS. "$E \in \mathcal{L}$."

Pf: $\forall \varepsilon_n = \frac{1}{n}$. $\exists$ closed set $F_n \subset E$ s.t $m^*(E - F_n) < \frac{1}{n}$

$F = \bigcup_{n \geq 1} F_n \qquad E \setminus F = E \setminus (\bigcup F_n) = \bigcap (E \setminus F_n) \subset E \setminus F_n$

$\therefore m^*(E - F) < m^*(E - F_n) < \frac{1}{n}$

$n \to \infty \quad m^*(E - F) = 0$. $G = E \setminus F$ is null set

$E = G \cup \underbrace{F}_{F_\sigma \in \mathcal{L}} \in \mathcal{L}$ $\square$

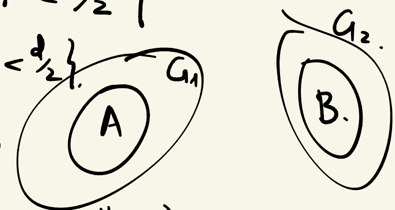
13 Similarly!

14. $A, B \subset \mathbb{R}^n$. If $d(A,B) = \inf_{\substack{x \in A \\ y \in B}} |x-y| > 0$.

WTS $m^*(A \cup B) = m^*(A) + m^*(B)$

Pf: $G_1 = \{ x : \operatorname{dist}(x, A) < d/2 \}$

$G_2 = \{ x : \operatorname{dist}(x, B) < d/2 \}$.



$G_1 \cap G_2 = \phi \Rightarrow A \cap B = \phi$

$\therefore m^*(A \cup B) = m^*(A) + m^*(B)$. $\square$

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15. $A, B \subseteq \mathbb{R}^n$. $m^*(A) + m^*(B) < +\infty$.

WTS $|m^*(A) - m^*(B)| \leq m^*(A \triangle B)$.

Pf: By definition $A \triangle B = (A-B) \cup (B-A)$

$$m^*(A) \leq m^*(B) + m^*(A-B)$$
$$\leq m^*(B) + m^*(A \triangle B).$$ $\square$

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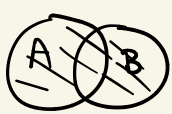
16. $A, B \in \mathcal{L}$. WTS: $m(A) + m(B) = m(A \cup B) + m(A \cap B)$.

Pf: $m(A) = m(A \cap B) + m(A \cap B^c)$

$m(B) = m(B \cap A) + m(B \cap A^c)$.

$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$



$\therefore m(A) + m(B) = m(A \cap B)$

$+ \underline{m(A \cap B^c) + m(A \cap B) + m(B \cap A^c)}$.

$= m(A \cup B)$

by 可数可加性

17 $E \in \mathcal{L}$. $x_0 \in \mathbb{R}^n$, 则 $E+\{x_0\} \in \mathcal{L}$, $m(E+\{x_0\}) = m(E)$.

Pf: WTS $E+\{x_0\}$. $\forall\, T \subset \mathbb{R}^n$.

$$m^*(T) = m^*(T-\{x_0\})$$
$$= m^*\Big(\underbrace{(T-\{x_0\})\cap E}_{T \,\cap\, (E+\{x_0\})}\Big) + m^*\big((T-\{x_0\})\cap E^c\big)$$

$\therefore E+\{x_0\} \in \mathcal{L}$. $\square$

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21. $\mathbb{R}^1$ 中作一个仅含无理数的闭集 F. s.t. $m(F)>0$

Pf: $\mathbb{R}$中有理数 $r_i \in \mathbb{Q}$, $G = \bigcup_{i\geq 1} \left(r_i - \frac{1}{2^i}, r_i + \frac{1}{2^i}\right)$

$m(G) \leq 1$. $F = \mathbb{R}^1 \setminus G$.

$m(F) > 0$. $\square$

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22 (1) $m(\underline{\lim}_{n\to\infty} E_n) \leq \underline{\lim}_{n\to\infty} m(E_n)$

(2) If $m(\bigcup_{n\geq 1} E_n) < +\infty$. WTS. $m(\overline{\lim}\, E_n) \leq \overline{\lim}\, m(E_n)$

Pf: $\underline{\lim}\, E_n = \bigcup_{k=1}^{+\infty} \bigcap_{n\geq k} E_n$ easy!

$\bigcap_{n\geq k} E_n = G_k$ $G_k \uparrow$, "$\lim m(\bigcup G_k) = m(\lim G_k)$".

That why "$m(\bigcup_{n\geq 1} E_n) < +\infty$" is necessary!

$\because \tilde{E}_n \downarrow$ $m(\lim \tilde{E}_n) = m(\bigcup \tilde{E}_n)$. $\square$

25. $E \in \mathcal{L}$ and $m(E) > 0$. WTS $\exists\, x \in E$. s.t. $\forall \delta > 0$.

we have $m(E \cap B(x, \delta)) > 0$.

Pf: $\exists\, \underset{\text{compact}}{K} \subset E$, $m(K) > 0$

证 $\exists\, x_0 \in K$, s.t $\forall \delta > 0$. $m(K \cap B(x_0, \delta)) > 0$.

反证: $\forall x_0 \in K$. $\exists\, \delta_0 > 0$. s.t $m(K \cap B(x_0, \delta_0)) = 0$.

$$K \subset \bigcup_{x_0 \in K} B(x_0, \delta_0).$$ By Heine-Borel theorem.

$$K \subset B(x_0^1, \delta_0^1) \cup B(x_0^2, \delta_0^2) \cup \cdots \cup B(x_0^N, \delta_0^N).$$

$$\Rightarrow K \subset \left(K \cap B(x_0^1, \delta_0^1)\right) \cup \cdots \cup \left(K \cap B(x_0^N, \delta_0^N)\right).$$

$m(K) \le \sum m(K \cap \sim) = 0$ Contradictory! $\square$

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28 $E \subset \mathbb{R}$, 且 $0 < a < m(E)$, WTS $\exists$ 无 inner points bonded closed set

$F \subset E$. & $m(F) = a$ Q: 与 cantor 集联系?

Remark: 在实轴上任意正测度集合，可找到"大小可控"的

$m = 0$ $\boxed{\text{无内点}}$ 有界闭子集 or 勒贝格测度可被"连续切割"成

想要的大小.

$f(x) = m((-\infty, x) \cap F)$ is a distribution function. C & $\uparrow$

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31. $f(x) \in C^1(\mathbb{R})$. $f'(x) > 0$. WTS if $E \subset \mathbb{R}$ is measurable.

we have $f^{-1}(E)$ is measurable set.

Pf: ① E is null set ② general case.

Rmk: $\forall E$, $\exists$ open cover $\{E_n\}$ s.t $E = \bigcup_{n\geq 1} E_n$, $f^{-1}(E) = \bigcup_{n\geq 1} f^{-1}(E_n)$

$\forall \varepsilon > 0$, $\exists I_n$: open interval. s.t $\bigcup I_n \supset E$ & $\sum |I_n| \leq \varepsilon$

$\forall E \subset [-M, M]$. $I_n \subset [-M, M]$ $|f'(x)| \geq \alpha > 0$. $\forall x \in f^{-1}([-M, M])$

$$|f^{-1}(I_n)| = |f^{-1}((a_n, b_n))| \leq |f^{-1\prime}(\xi)||b_n - a_n| \leq \frac{1}{\alpha}|b_n - a_n| = \frac{1}{\alpha}|I_n|$$

$$\therefore \sum_{n\geq 1} |f^{-1}(I_n)| \leq \frac{1}{\alpha}\sum_{n\geq 1}|I_n| < \frac{1}{\alpha}\varepsilon$$

$\because f^{-1}(E) \subset f^{-1}(\bigcup I_n)$. $\therefore |f^{-1}(E)| \leq \frac{1}{\alpha}\varepsilon \longrightarrow 0$. ✓

② $E = \underbrace{\bigcap G_n}_{\text{open set}} \setminus \underbrace{E_0}_{\text{null set}}$ $\qquad f^{-1}(E) = f^{-1}(\bigcap G_n) \setminus f^{-1}(E_0)$

$$= \bigcap \underbrace{f^{-1}(G_n)}_{\text{open set}} \setminus f^{-1}(E_0)$$

□

Q: 每个闭集必是可数开集的交集.

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32. if $E \subset \mathbb{R}^n$, $\exists$ G_δ set $H \supset E$, s.t $m(H) = m^*(E)$.

we say, H是E的等测包.

Pf: $E \subset \bigcup_{n\geq 1} I_n$ s.t $\sum |I_n| \leq m^*(E) + \frac{1}{n}$

Let $G_n = \bigcup_{n\geq 1} I_n$. $\therefore m^*(E) \leq m^*(G_n) \leq m^*(E) + \frac{1}{n}$.

Take $\underbrace{G}_{G_\delta \text{ set}} = \bigcap_{n\geq 1} G_n \implies m^*(E) \leq m^*(G) \leq m^*(E) + \frac{1}{n}$

$m^*(H) = m^*(E)$ □

Chap 4.

9. (Lusin Thm's inverse) If $\forall \delta > 0$, $\exists$ $F_\delta \subset E$ (closed) s.t. $m(E \setminus F_\delta) \le \delta$ and $f(x)$ is continuous on F_δ, WTS $f(x) \in \mathcal{L}(E)$. (measurable.)

Pf: $\forall n$, $\exists$ $F_n \subset E$ (closed) s.t. $m(E \setminus F_n) \le \frac{1}{n}$

$f \in C(F_n)$ Define $F = \bigcup_{n \ge 1} F_n$ $\quad m(E \setminus F) \le m(E \setminus F_n) \le \frac{1}{n}$

$\therefore m(E \setminus F) = 0$. F is measurable. $[E \supset E_0 = E \setminus F, \quad m(E_0) = 0]$

$\forall a \in \mathbb{R}$. $\underbrace{\{x \mid f(x) > a\}}_{= E \cap \{x \mid f(x) > a\}}$ ($\subset E$) $= (E_0 \cap \{x \mid f(x) > a\}) \cup (\underline{\bigcup F_n} \cap \{x \mid f(x) > a\})$

$\iff E[f > a] = (E_0 \cap E[f > a]) \cup \left(\bigcup_{n=1}^{\infty} \{x \in F_n \mid f(x) > a\} \right)$

$= \underbrace{(E_0 \cap E[f > a])}_{\text{null set}} \cup \underbrace{\left(\bigcup_{n=1}^{\infty} F_n[f(x) > a] \right)}_{\because f \in C(F_n) \therefore \text{measurable.}}$

$\square$

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10. $f_n \xrightarrow{m} f$ on E, & $f_n(x) \le g(x)$ a.e. on E. WTS. $f(x) \le g(x)$ a.e. on E

Pf $f_n \xrightarrow{m} f$ on E $\iff$ $m(\{x \mid |f_n - f| > \varepsilon\}) < \varepsilon$ for any $\varepsilon > 0$,

$\Downarrow$

Riesz. thm. $\exists$ a subsequence of (f_n), (f_{n_j}) s.t.

$f_{n_j} \longrightarrow f$ a.e. on E $\quad \exists E_1 \subset E$. s.t. $m(E_1) = 0$

$f_{n_j}(x) \longrightarrow f$ on $E \setminus E_1$.

$\exists E_2 \subset E$. $f_n(x) \le g(x)$ on $E \setminus E_2$ where $m(E_2) = 0$

Take $E_0 = E_1 \cup E_2$. $m(E_0) = 0$.

$\forall x \in E \setminus E_0$. (1) $f_{n_j}(x) \to f$

(2) $f_n(x) \le g(x) \Rightarrow \underset{\downarrow\, f}{f_{n_j}} \le g$, $j \to +\infty$

$\square$

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11 $\underline{f_n \rightrightarrows f \text{ on } E}$, $\underline{f_n \le f_{n+1} \text{ a.e. on } E}$. WTS $f_n \to f$ a.e.

<u>Pf</u>: $\because$ $\therefore \exists E_1 \subset E$, $m(E_1) = 0$.

s.t $f_n \le f_{n+1}$ on $E \setminus E_1$

$\because (f_n) \nearrow$ on $E \setminus E_1$. Take $\lim\limits_{n \to +\infty} f_n = \tilde{f}$ on $E \setminus E_1$

$\exists$ subsequence. (f_{n_j}) $f_{n_j} \to f$ a.e.

$\exists E_2 \subset E$. $m(E_2) = 0$. $f_{n_j} \to f$ on $E \setminus E_2$.

Take $E_0 = E_1 \cup E_2$. on $E \setminus E_0$, we have.

(1) $(f_n) \nearrow$ $\lim\limits_{n \to \infty} f_n = \tilde{f}$

(2) $f_{n_j} \to f$ $\therefore f = \tilde{f}$ $\forall x \in E \setminus E_0$

$\Rightarrow f_n(x) \to f$ $\forall x \in E \setminus E_0$ $\square$

<u>12</u>. f_n f on E, $f_n(x) = g_n(x)$ a.e. WTS $g_n(x)$ $f(x)$

<u>Pf</u>: $m(\{x : |g_n(x) - f(x)| > \delta\})$

$= m(\{x : |f_n(x) - g_n(x)| > \delta\}) \longrightarrow 0$.

$\square$

<u>13</u>. $m(E) < +\infty$ WTS. $f_n \to f$ on $E \iff \forall f_{n_k}.\ \exists f_{n_{k_j}}$

s.t. $\lim\limits_{j \to \infty} f_{n_{k_j}} = f(x)$ a.e. E

17. $m(E) < +\infty$. (f_n) measurable. WTS. $\lim\limits_{n\to+\infty} f_n(x) = 0$ a.e. on E.

$\iff$ $\forall \varepsilon > 0$. we have $\lim\limits_{n\to\infty} m\left(\{x \in E \mid \sup\limits_{k\geq n} |f_k(x)| \geq \varepsilon\}\right) = 0$.

Pf: $(\Longrightarrow)$ By Egorov thm. $\forall \delta > 0$. $\exists$ $E_\delta \subset E$, s.t.

$m(E \setminus E_\delta) < \delta$. $f_n \rightrightarrows f$ on E_δ.

$\forall \varepsilon > 0$. $\exists N$. s.t $\forall k > N$. $|f_k - f| < \varepsilon$. $\iff |f_k| < \varepsilon$ on E_δ

$\iff \sup\limits_{k\geq n} |f_k| < \varepsilon$. on E_δ. $\Rightarrow$ on $E \setminus E_\delta$. $\sup\limits_{k\geq n} |f_k| \geq \varepsilon$.

$E_k = \{x : \sup |f_k| \geq \varepsilon\} \subset E \setminus E_\delta$. $\lim\limits_{n\to\infty} m(E_k) = m(E \setminus E_\delta) = 0$.

$(\Longleftarrow)$ If $x \notin E_k$. $\sup\limits_{k\geq n} |f_k| < \varepsilon$.

$E[|f_n| \geq \varepsilon] \subset E_k$.

$m(\sim) \leq \lim m(E_k) = 0$. $\square$

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18. f_n. f 是$[a,b]$上几乎处处有限的可测函数. 且 $\lim\limits_{n\to\infty} f_n(x) = f(x)$ a.e.

试证明. $\exists$ $E_n \subset [a,b]$ s.t $m\left([a,b] \setminus \bigcup E_n\right) = 0$.

且 $f_n(x) \rightrightarrows f(x)$ on E_n $\forall n$.

Pf: By Egorov thm. $\exists E_n \subset [a,b]$ s.t $m([a,b] \setminus E_n) < \frac{1}{n}$

$f_n(x) \rightrightarrows f(x)$ on E_n.

Take $E = \bigcup\limits_{n\geq 1} E_n$ $m([a,b] \setminus E) \leq m([a,b] \setminus E_n) < \frac{1}{n}$

$\square$

Chap 5

1. $f(x) > 0$ a.e. on E 可测 且满足 $\int_E f(x)dx = 0$. 试证明 $m(E) = 0$.

Pf: (反证) If not. $m(E) \neq 0$. $E = \underbrace{E[f \leq 0]}_{m(\cdot)=0} \cup \left(\bigcup_{n \geq 1} E(\frac{1}{n} \leq f] \right)$

$\exists n_0 \in \mathbb{N}$, s.t. $m\left(E(\frac{1}{n_0} \leq f)\right) > 0$.

$\therefore 0 = \int_E f(x)dx \geq \int_{E[f \geq \frac{1}{n_0}]} f dx \geq \frac{1}{n_0} m\left(E[\frac{1}{n_0} \leq f]\right) > 0$. Contradiction!

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2. $f: E$ 上可积. $e_n = E(|f| \geq n)$. 则 $\lim_{n \to +\infty} n\, m(e_n) = 0$

"高处不胜寒!"

Pf: $\int_{e_n} f dx = n \cdot m(e_n)$

$\lim_{n \to \infty} \int_{e_n} f dx = 0$. □

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3. $f: E$ 上非负可积. Let $E_k = \{x \in E : f(x) > k\}$. WTS $\sum_{k=1}^{\infty} m(E_k) < \infty$

Proof: $\int_{E_k} f(x) \geq \int_{E_k} k = k\, m(E_k)$

$m(E_k) \leq \frac{1}{k} \int_{E_k} f(x)$

$\sum_{k=1}^{+\infty} m(E_k) \leq \sum \int_{E_k} \frac{f(x)}{k}$

$E_1 \supset E_2 \supset \cdots$

↓ decomposition

$F_k = \{x \in E : k \leq f(x) < k+1\}$.

$\boxed{F_i \cap F_j = \emptyset \text{ if } i \neq j}$

$F_k = E_k \cap E_{k+1}^c$ $\quad E = \bigsqcup F_k$.

$+\infty > \int_E f = \int_{\sqcup F_k} f = \sum_{k=0}^{+\infty} \int_{F_k} f \geq \sum_{k=0}^{+\infty} k \cdot m(F_k)$

$= \sum m(E_k)$ □

4. $m(E) < +\infty$, $f(x)$: E上可测, $E_n = E[n-1 \le f < n]$. 则 f 在E上可积 $\Longleftrightarrow$ $\sum_{-\infty}^{+\infty} |n|\, mE_n < \infty$

Pf: ($\Rightarrow$) f 在E上可积 $\Rightarrow$ $\exists\, E_1 \subset E$, $m(E_1) = 0$.

s.t. $f < +\infty$ a.e. on $E \setminus E_1$.

$$\therefore\ E \setminus E_1 = \bigcup_{-\infty}^{+\infty} E_n \qquad m(E) = m\left(\bigcup E_n\right)$$

$$+\infty > \int_{E\setminus E_1} |f| = \int_{\bigcup_{-\infty}^{+\infty} E_n} |f| = \sum_{-\infty}^{+\infty} \int_{E_n} |f| = \underbrace{\sum_{-\infty}^{0} \int_{E_n} |f|}_{\sum_{-\infty}^{0} n\, m(E_n)} + \underbrace{\sum_{0}^{+\infty} \int_{E_n} |f|}_{\sum_{1}^{\infty} (n-1) m(E_n)}$$

$$\ge \sum_{1}^{\infty} (n-1) m(E_n) + \sum_{-\infty}^{0} n\, m(E_n)$$

$$= \sum_{-\infty}^{+\infty} |n|\, m(E_n) - \sum_{1}^{\infty} m(E_n)$$

$$\therefore\ \sum_{-\infty}^{+\infty} |n|\, m(E_n) \le \underbrace{\sum_{n=1}^{+\infty} m(E_n)}_{\le\, m(E)} + \int_{E\setminus E_1} |f| < +\infty. \quad \checkmark$$

($\Leftarrow$) Similarity! $\square$

5. f 在$[a,b]$上R反常积分存在, WTS: f 在$[a,b]$上L可积 $\Longleftrightarrow$ $|f|$ 在$[a,b]$上R ------. 并证明此时 $\int_{[a,b]} f\,dx = (R)\int_a^b f(x)\,dx$

6. $\{f_n\}$: E上非负可积. If $\lim\limits_{n\to\infty}\int_E f_n\,dx = 0$, 则 $f_n(x) \xrightarrow{m} 0$

$\forall \delta>0,\ m(\{f_n(x) > \delta\}) < \delta$

Pf:

$\forall \sigma > 0.\ \int_E f_n\,dx \geq \int_{E[f_n>\sigma]} f_n\,dx$

$$\geq \sigma\, m(E[f_n(x) > \sigma])$$

$$\Rightarrow m(\{f_n > \delta\}) \leq \frac{1}{\sigma}\int_E f_n\,dx = 0.$$

$\lim\limits_{n\to\infty}$

$\therefore f_n(x) \xrightarrow{m} 0$ □

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8. 设 $f, f_k \in \mathcal{L}(E)$ $(k=1,2,\dots)$ 且 $\lim\limits_{k\to\infty}\int_E |f_k - f|\,dx = 0$

WTS. $\{f_k(x)\}$ 必有子列 $\{f_{k_i}(x)\}$ s.t. $\lim\limits_{i\to\infty} f_{k_i} = f$ a.e. $x\in E$.

Pf:

$\forall \varepsilon > 0.\ \forall \delta > 0.\ \exists N_1 \in \mathbb{N}.$ s.t $k > N$, we have.

$$m(\{x : |f_k - f| > \varepsilon\}) \leq \frac{1}{\varepsilon}\int_E |f_k - f| < \delta \Rightarrow f_k \xrightarrow{m} f$$

According to the Riesz thm

$\exists$ subsequence. $f_{k_i} \to f$ □

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9. f: real-value function on $\mathbb{R}^n$. $\forall \varepsilon > 0$. $\exists$ L-可积. g & h.

s.t. $g(x) \leq f(x) \leq h(x)$ $x \in \mathbb{R}^n$. & $\int_{\mathbb{R}^n} |h(x) - g(x)|\,dx < \varepsilon$.

WTS. $f(x)$ 在 $\mathbb{R}^n$ 上 L-可积.

Pf: $\forall \delta > 0$. $\exists\, g_n, h_n$ s.t $g_n \le f \le h_n$.

& $\int_{\mathbb{R}^n} |h_n - g_n| dx < \frac{1}{n}$

& $m(\{x : |h_n - g_n| \geq \delta\}) \le \frac{1}{\delta} \int_{\mathbb{R}^n} |h_n - g_n| dx < \frac{1}{n\delta} \to 0$

$\therefore\ h_n - g_n \xrightarrow{m} 0.$

By Riesz thm. $\exists$ subsequence. $h_{n_k} - g_{n_k} \longrightarrow 0$ a.e.

$\therefore \lim_{k\to\infty} h_{n_k} = \lim_{k\to\infty} g_{n_k} = f(x)$ a.e.

$\Rightarrow f$ 可测. $\therefore \exists\, g_n, h_n$ 可积.

s.t $g_n(x) \le f(x) \le h_n(x)$

故 $|f(x)| \le \underbrace{|g_n(x)| + |h_n(x)|}_{\text{可积}}$ $\therefore f$ 可积. $\square$

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11. 设 $f(x) = (\sin \frac{1}{x}) / x^{\alpha}$, $0 < x \le 1$. 讨论 $\alpha = ?$ $f(x)$ 在 $[0,1]$ 上 L可积

or 不可积.

13 $f:[0,1]$ 可积. 若 $\forall c\ (0<c<1)$ 恒有 $\int_0^c f(x)dx=0$

则 $f\sim 0$.

Pf: $\int_0^1 f\,dx=0$ (why?)

$\forall (\alpha,\beta)\subset(0,1)$

$$\int_\alpha^\beta f=\int_0^\beta f-\int_0^\alpha f=0$$

$\Rightarrow \forall$ open set $G=\bigcup_{i\geq 1}(\alpha_i,\beta_i)$

$$\int_G f=\sum_i\int_{(\alpha_i,\beta_i)} f=0$$

$\therefore \forall$ closed set $F\subset(0,1)$.

$G=(0,1)-F$ is open

$$\int_F f(x)=\int_{(0,1)} f-\int_G f=0$$

若 $E=(0,1)$. $f\nsim 0$

$E_1=E[f>0]$

$E_2=E[f<0]$

至少一个 m 大于 0

不妨设 $mE_1>0$

$\therefore \exists F_0$ closed set $\subset E_1$

s.t. $m(F_0)>0$

$f(x)>0$

$\Rightarrow \int_{F_0} f>0$ 矛盾! □

14. 设 $m(E)\neq 0$. $f:E$ 上可积. 若 $\forall$ 有界可测 $\varphi(x)$ $\int_E f(x)\varphi(x)=0$.

则 $f=0$ a.e. on E.

Pf: $\varphi(x)=\begin{cases}1 & f(x)\geq 0\\ -1 & f(x)<0\end{cases}$ $\int_E f(x)\varphi(x)dx=\int_E |f(x)|dx=0$

$\Rightarrow f=0$ a.e. on E.

15. 设$f \in \mathcal{L}(R^n)$，若对一切R^n上具有紧支集的连续函数$\varphi(x)$，均有$\int_{R^n} f(x)\varphi(x)dx = 0$，则$f(x) = 0 \quad a.e. x \in R^n$.

Pf: If not. "$f(x) > 0$" $(\varphi_k(x))$

s.t. $\lim_{k\to\infty} \int |\chi_E(x) - \varphi_k(x)| dx = 0$

$|\varphi_k(x)| \le 1$, $\lim_{k\to\infty} \varphi_k = \chi_E$ a.e.

$\because |f(x)\varphi_k(x)| \le |f(x)|$

$\therefore 0 < \int_E f(x)dx = \int_{R^n} f(x)\chi_E dx$

$= \lim_{k\to\infty} \int f(x)\varphi_k(x) = 0$，矛盾! □

16. 设$\{f_n(x)\}$是E上一列非负可测函数，并且依测度收敛于$f(x)$.试证明

$$\int_E f(x)dx \le \varliminf_{n\to\infty} \int_E f_n(x)dx$$

Pf: $f_n \ge 0$. measurable. $f_n \xrightarrow{m} f(x)$. which means $m(\{x : |f_n - f| > \delta\}) < \delta$ for $\forall \delta > 0$

$\exists$ subsequence $f_{n_j} \longrightarrow f$ a.e.

By Fatou's Lemma $\int_E f\,dx \le \varliminf_{j\to\infty} \int_E f_{n_j} dx = \varliminf_{n\to\infty} \int_E f_n dx$ □

17. 设$F \in L(E), f_k \in L(E) (k \in N)$若有

$$f_k(x) \ge F(x) \quad (x \in E), \quad \varliminf_{k\to\infty} \int_E f_k(x)dx \ne +\infty,$$

则

$$\int_E \varliminf_{k\to\infty} f_k(x)dx \le \varliminf_{k\to\infty} \int_E f_k(x)dx$$

对比本有
给出18题解答

18. 设$f_k \in L(E)(k \in N), F \in L(E)$。若有

$$f_k(x) \le F(x) \quad (x \in E), \quad \overline{\lim_{k\to\infty}} \int_E f_k(x)dx \ne -\infty,$$

则$\overline{\lim_{k\to\infty}} f_k(x)$在$E$上可积，且有

$$\int_E \overline{\lim_{k\to\infty}} f_k(x)dx \ge \overline{\lim_{k\to\infty}} \int_E f_k(x)dx$$

Pf: $\forall n \in \mathbb{N}$. $F(x) \geqslant \sup_{k\geqslant n}\{f_k(x)\} \geqslant f_k(x)$.

$\therefore \int_E f_k \le \int_E \sup f_k \le \int F(x)$

$-\infty < \overline{\lim_{k\to\infty}} \int_E f_k \le \int_E \sup f_k < +\infty$

Notice ~~that~~ $\sup_{k\geqslant n} f_k \geqslant \sup_{k\geqslant n+1} f_k \quad \therefore n \to +\infty$

$\overline{\lim_{k\to\infty}} \int_E f_k \le \lim_{n\to\infty} \int_E \sup_{k\geqslant n} f_k = \int_E \lim_{n\to\infty} \sup_{k\geqslant n} f_k = \int_E \overline{\lim} f_k < +\infty$

$\square$

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21. 证明：

$$\lim_n \int_0^\infty \frac{dt}{(1+\frac{t}{n})^n t^{1/n}} = 1.$$

$$\lim_{n\to\infty} \left[\left(1+\frac{t}{n}\right)^{n/t}\right]^t = e^t$$

Pf: $\lim_{n\to\infty} \int_0^\infty \frac{dt}{(1+\frac{t}{n})^n t^{1/n}} = \int_0^\infty \lim_{n\to\infty} \frac{dt}{(1+\frac{t}{n})^n t^{1/n}}$

$= \int_0^\infty \frac{dt}{e^t} = 1.$

Why?

By DCT $\because$ $\frac{1}{(1+\frac{t}{n})^n t^{1/n}} \le F(t) \quad n \geqslant 2.$

$F(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0<t\le 1 \\ 4t^{-2} & t>1 \end{cases}$

可积.

$\left\{\begin{array}{l} (1+\frac{t}{n})^n t^{1/n} \geqslant \sqrt{t} \quad 0<t\le 1 \\ \sim \quad \geqslant \frac{n(n-1)}{2}\left(\frac{t}{n}\right)^2 \\ \quad \geqslant \frac{1}{4}t^2 \end{array}\right.$

$\square$

22. 证明

$$\lim_{n\to\infty}\int_0^{+\infty}\frac{(\ln(x+n))^p}{n}\exp^{-x}\cos x dx=0,$$

其中p为固定的正数.

Pf: $\displaystyle\lim_{n\to\infty}\int_0^{+\infty}\frac{\ln^p(x+n)}{\underbrace{n}_{\to 0}}\exp^{-x}\cos x\,dx=0$

$\displaystyle\lim_{x\to+\infty}\frac{(\ln x)^p}{x}=0 \quad \therefore \exists M>0,\ \text{s.t.}\ \left|\frac{\ln^p(x+n)}{x+n}\right|\le M.$

$$\left|\frac{\ln^p(x+n)}{n}\exp^{-x}\cos x\right|\le\left|\frac{\ln^p(x+n)}{x+n}\left(1+\frac{x}{n}\right)\exp^{-x}\cos x\right|$$

$$\le M(1+x)\exp(-x)=F(x).$$

$\int_0^{+\infty}F(x)<+\infty.$ $\therefore$ By DCT. Green part is right! $\square$

23. 设$\{f_n\}$为E上可积函数列，$\lim\limits_{n\to\infty}f_n(x)=f(x)$a.e. 于$E$，且$\int_E|f_n(x)|dx<K$, 其中$K$为常数，则$f(x)$可积.

Pf: By Fatou's Lemma

$$\int_E|f(x)|dx=\int_E\lim_{n\to\infty}|f_n(x)|\,dx\le\varliminf_{n\to\infty}\int_E|f_n(x)|\,dx<K<+\infty.$$

$\square$

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24. 设$f\in\mathcal{L}(R)$, $f(0)=0$且$f'(0)$存在有限.证明$\int_{R^1}\frac{f(x)}{x}dx$存在有限.

Pf:

$$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{f(x)}{x}<+\infty.$$

$\forall\varepsilon>0,\exists\delta>0,\ 0<|x|<\delta$ 时，有 $\left|\frac{f(x)}{x}-f'(0)\right|<\varepsilon=1$

$f'(0)-1<\frac{f(x)}{x}<f'(0)+1,\quad \left|\frac{f(x)}{x}\right|\le\max\{f'(0)-1,\ f'(0)+1\}$

$=M.$

$$\int_{R^1}\left|\frac{f(x)}{x}\right|dx = \int_{\delta}^{+\infty}\left|\frac{f(x)}{x}\right|dx + \underline{\int_0^{\delta}\left|\frac{f(x)}{x}\right|dx}$$

$$+ \underline{\int_{-\delta}^{0}\left|\frac{f(x)}{x}\right|dx} + \int_{-\infty}^{-\delta}\left|\frac{f(x)}{x}\right|dx$$

$$< \underline{2M\delta} + \frac{1}{\delta}\int_{R^1}|f(x)|dx < +\infty \qquad \therefore \frac{f(x)}{x} \in \mathcal{L}. \quad \square$$

---

27. 设$f(x,t)$当$|t-t_0|<\delta$时为x的在$[a,b]$上的可积函数，又有常数K，使

$$\left|\frac{\partial}{\partial t}f(x,t)\right| \le K,\ a \le x \le b,\ |t-t_0| < \delta,$$

则

$$\frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b f_t'(x,t)dx.$$

Pf: Define $g(t) = \int_a^b f(x,t)\,dx$

$$g'(t) = \lim_{n\to\infty}\frac{g(t+\frac{1}{n}) - g(t)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\int_a^b[f(x,t+\frac{1}{n}) - f(x,t)]dx}{1/n}$$

$$= \lim_{n\to\infty}\frac{\int_a^b \frac{\partial f}{\partial t}(x,\xi)\frac{1}{n}\,dx}{\frac{1}{n}} = \int_a^b f_t'(x,t)\,dt$$

$\because |f_t'| \le K$, By DCT.

$$\lim\int f_n = \int_a^b \underline{\lim f_n} = \int_a^b f_t' \quad \checkmark \quad \square$$

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29. 在$D: -1 \le x \le 1,\ -1 \le y \le 1$上定义

$$f(x,y) = \begin{cases} \dfrac{x\cdot y}{(x^2+y^2)^2}, & x^2+y^2 \neq 0, \\ 0, & x = y = 0. \end{cases}$$

则$f(x,y)$的两个累次积分存在且相等，但$f(x,y)$在D上不可积分.

Pf: $\exists f$ $f \in \mathcal{L}(D)$. $\Rightarrow |f| \in \mathcal{L}(D) \Rightarrow \iint |f| < +\infty$.

But $\int_{-1}^{1}dy\int_{-1}^{1}|f|\,dx = 2\int_{-1}^{1}dy\int_0^1 |y|\frac{x}{(x^2+y^2)^2}dx \overset{\cdots}{=} +\infty$. $\square$

30. 设$f(x),g(x)$是E上非负可测函数且$f(x)g(x)$在E上可积. 令$E_y=E[g\geq y]$。证明：

$$F(y)=\int_{E_y}f(x)dx$$

对一切$y>0$都存在，且成立

$$\int_0^{+\infty}F(y)dy=\int_E f(x)g(x)dx.$$

Pf: (1) $E_y\subset E$

$$\int_{E_y} f\cdot g\leq\int_E f\cdot g<+\infty$$

$$\int_{E_y} y f(x)dx \qquad \therefore \int_{E_y} f(x)dx<+\infty. \quad F(y) \text{ 有}~~\text{意}~~\text{义}.$$

(2) $$\int_0^{+\infty}F(y)dy=\int_0^{+\infty}dy\int_{E_y}f(x)dx=\int_0^{+\infty}dy\int_E\chi_{E_y}f(x)dx$$

$$=\int_E f(x)dx\int_0^{+\infty}\chi_{E_y}dy=\int_E f(x)dx\int_0^{g(x)}1\,dy$$

$$=\int_E f(x)g(x)dx$$

□

32. 设$mE<\infty$,且$\{f_n(x)\}$是E上的非负可积函数列，且有$f\in\mathrm{L}(E)$，使得$f_n(x)$在E上依测度收敛于$f(x)$。若

$$\lim_{n\to\infty}\int_E f_n(x)dx=\int_E f(x)dx$$

试证明

$$\lim_{n\to\infty}\int_E|f_n(x)-f(x)|dx=0$$

Pf: $f_n\xrightarrow{m}f \Rightarrow$ By Riesz thm.

$\exists\, f_{n_j}\to f$ a.e. on E

$\because f\in L(E)$ $\therefore \forall\varepsilon>0, \exists\delta, A\subset E, m(A)<\delta.$ $\int_A|f|dx<\varepsilon/3$

By Egorov thm. $\exists E_0\subset E.$ $m(E\setminus E_0)<\delta.$

$f_{n_j}\rightrightarrows f$ on E_0. $\therefore \exists N_1.$. $j>N_1$. we have.

$$|f_{n_j}-f|<\frac{\varepsilon}{3m(E)}$$

$$\int_E f_{n_j}dx\longrightarrow\int_E f\,dx \qquad \int_{E_0}f_{n_j}dx\to\int_{E_0}f\,dx \qquad \int_{E_0}|f_{n_j}-f|dx\to 0$$

$\because \int_{E\setminus E_0} f_{n_j} \to \int_{E\setminus E_0} f \quad \therefore \exists N_2$ s.t. $j > N_2$.

$$\left| \int_{E\setminus E_0} f_{n_j} - \int_{E\setminus E_0} f \right| < \varepsilon/3$$

$$\therefore \int_{E\setminus E_0} |f_{n_j}|\,dx < \varepsilon/3 + \int_{E\setminus E_0} |f| < \frac{2}{3}\varepsilon.$$

$\because \int_{E_0} |f_{n_j} - f|\,dx \to 0$. $\exists N_3$. $j > N_3$ $\int_{E_0} |f_{n_j} - f|\,dx < \varepsilon/3$

$\therefore$ when $j > N = \max\{N_1, N_2, N_3\}$

$$0 \leftarrow \int_E |f_{n_j} - f|\,dx \le \int_{E_0} |f_{n_j} - f|\,dx + \int_{E\setminus E_0} |f_{n_j}|\,dx + \int_{E\setminus E_0} |f|\,dx \le \frac{4\varepsilon}{3}$$

$\therefore \forall f_{n_j} \ \exists f_{n_{j_k}}$ s.t. $\int_E |f_{n_{j_k}} - f|\,dx \to 0$. $(k \to \infty)$.

$\Longrightarrow$ By Heine Thm. $\int_E |f_n - f|\,dx \to 0$ $\square$

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33. 设$f(x)$在$[a-\epsilon, b+\epsilon]$上可积分，则

$$\lim_{t\to 0} \int_a^b |f(x+t) - f(x)|dx = 0.$$

找 $\varphi(x) \in [a-\varepsilon, b+\varepsilon]$ $\forall \varepsilon > 0$, $\forall \delta > 0$.

$$\int_{a-\varepsilon}^{b+\varepsilon} |f(x) - \varphi(x)|\,dx \le \delta/3$$

$$\int_a^b |f(x+t) - f(x)|\,dx \le \int_a^b |f(x+t) - \varphi(x+t)|\,dx + \int_a^b |\varphi(x+t) - \varphi(x)|\,dx + \int |\varphi(x) - f(x)|\,dx$$

$\square$

34. 若f在R上L-可积，证明$\int_R |f(x+h)-f(x)|dx \to 0, h \to 0$.

Pf: $\because$ f L-可积，$\forall \varepsilon>0$. $\exists R>0$. s.t

$$\int_{|x|\geq R} |f| dx < \varepsilon/3. \quad \text{(远处很小)}$$

$$\int_{-\infty}^{+\infty} |f(x+h)-f(x)|dx = \int_{|x|\geq R} |f(x+h)-f(x)|dx + \int_{|x|<R} |f(x+h)-f(x)|dx$$

$$\leq 2\int_{|x|\geq R} |f| dx + \int_{|x|<R} |f(x+h)-f(x)|dx$$

$\searrow h \to 0$

$$\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3}.$$

$$= \varepsilon. \qquad \square$$

35. 设$f(x)$是E上的可测函数，对任意的$\lambda>0$，作点集$\{x\in E:|f(x)|>\lambda\}$，它是可测集，我们称

$$f_*(\lambda) = m(\{x \in E : |f(x)| > \lambda\})$$

为f的分布函数，证明$(1 \leq p \leq \infty)$

$$\int_E |f(x)|^p dx = p\int_0^\infty \lambda^{p-1} f_*(\lambda) d\lambda$$

Pf: $p\int_0^\infty \lambda^{p-1} m(\{x\in E: |f(x)|>\lambda\}) d\lambda$

$$= p\int_0^\infty \lambda^{p-1}\Big(\int_E \chi_{\underbrace{\{x\in E: |f(x)|>\lambda\}}_{=: F}} dx\Big) d\lambda$$

Fubini's theorem

$$= p\int_E dx \int_0^\infty \chi_F \lambda^{p-1} d\lambda$$

But! $\chi_F \lambda^{p-1}$ 可积性？

$$= p\int_E dx \int_0^{|f(x)|} \lambda^{p-1} d\lambda$$

$$= \int_E |f(x)|^p dx \qquad \square$$

---

36. 设$f(x)$是$(0,1)$上的非负可测函数，若存在常数c使得

$$\int_0^1 [f(x)]^n dx = c, \quad n = 1, 2, \cdots$$

试证明存在可测集$E \subset (0,1)$，使得$f(x) = \chi_E(x)$, $a.e. x \in [a,b]$.

37. 设$f(x), f_n(x)$ $(n=1,2,\cdots)$都是E上的可积函数，$\underline{\lim\limits_{n\to\infty} f_n(x) = f(x)}$a.e. 于$E$，且 (∗∗)

$$\underline{\lim_{n\to\infty}\int_E |f_n(x)|dx = \int_E |f(x)|dx,} \quad (*)$$

试证，在任意可测子集$e\subset E$,都有

$$\lim_{n\to\infty}\int_e |f_n(x)|dx = \int_e |f(x)|dx.$$

Pf: Fatou's Lemma

$$\int_e \liminf_{n\to\infty} |f_n(x)|\,dx = \int_e |f_n(x)|\,dx \le \liminf_{n\to\infty} \int |f_n(x)|\,dx$$

$$\overset{\int_{E\setminus e}|f(x)|dx}{\overset{\shortparallel}{\int_{E\setminus e}}} \underbrace{\liminf_{n\to\infty} |f_n(x)|}_{(**)} \le \liminf_{n\to\infty}\int_{E\setminus e} |f_n(x)|\,dx$$

$$\because \int_e |f_n(x)|\,dx = \int_E |f_n(x)|\,dx - \int_{E\setminus e} |f_n(x)|\,dx$$

$$\limsup_{n\to\infty}\int_e |f_n(x)| \le \underline{\limsup \int_E |f_n(x)|\,dx} - \liminf \int_{E\setminus e} |f_n(x)|\,dx$$

$$\overset{(*)}{=} \underline{\int_e |f(x)|\,dx + \int_{E\setminus e} |f(x)|\,dx} - \liminf \int_{E\setminus e} |f_n(x)|\,dx$$

$$\le \int_e |f(x)|\,dx$$

$$\therefore \int_e |f(x)|\,dx \underset{\text{“}=\text{”}}{\le} \lim_{n\to\infty} \inf \int_e |f_n(x)|\,dx$$

$$\underset{\text{“}=\text{”}}{\le} \limsup_{n\to\infty} \int_e |f_n(x)|\,dx \underset{\text{“}=\text{”}}{\le} \int_e |f(x)|\,dx \qquad \square$$

38. 设$f(x)$是E上的几乎处处有限的非负可测函数，$m(E)<\infty$。在$[0,\infty)$上作如下划分

$$0=y_0<y_1<\cdots<y_k<y_{k+1}<\cdots\to\infty$$

其中$y_{k+1}-y_k<\delta(k=0,1,\cdots)$。若令

$$E_k=\{x\in E: y_k\le f(x)<y_{k+1}\},\quad k=0,1,\cdots$$

则$f(x)$在E上是可积的，当且仅当级数

$$\sum_{k=0}^{\infty}y_k m(E_k)<\infty$$

此时有

$$\lim_{\delta\to 0}\sum_{k=0}^{\infty}y_k m(E_k)=\int_E f(x)dx$$

Pf: $(\Longrightarrow)$

$$\int_E f = \sum_k \int_{E_k} f$$

$$\geq \sum_k y_k m(E_k)$$

$\because f\in\mathcal{L}(E)\Rightarrow \int_E f<+\infty$

$\therefore \sum_k y_k m(E_k)<+\infty$

Define $f_\delta=\sum y_k \chi_{E_k}$

$$\int_E (f-f_\delta)dx = \int_E f-\sum_k y_k m(E_k)$$

If $x\in E$, $x\in E_k$. $\Rightarrow$ $f_\delta \geq y_k$

$\therefore$ $f-f_\delta \leq f-y_k \longrightarrow 0$ $\quad \delta\to 0$

$\because f-f_\delta\leq f$

By DCT $\int_E (f-f_\delta)dx \longrightarrow 0$ $\square$

40. 设$f \in L(R^n)$,且满足$\int_{R^n} f(x)dx = c > 0$。试证明对$\lambda \in (0,c)$，存在$E \subset R^n$，使得

$$\int_E f(x)dx = \lambda$$

Pf: $g(t) = \int_{|x| \leq t} f(x)\,dx.$ $g \in C.$ $g(0) = 0.$

$g(\infty) = c.$ 介值定理 √

41. 设$f(x)$是E上的有界可测函数，且存在正数M以及$\alpha < 1$，使得对于任意的$\lambda > 0$，有

$$m(\{x \in E : |f(x)| > \lambda\}) < \frac{M}{\lambda^\alpha},$$

试证明$f \in \mathcal{L}(E)$.